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Existence of Solutions for Neutral Functional Differential Equations with Causal Operators

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1. INTRODUCTION

$$\frac{d}{dt}(Vx)(t) = (Wx)(t), \qquad t \in [0, T], \tag{1}$$

in which V and W are acting on a function space $E([0, T], R^n)$, which is chosen among the spaces of continuous or measurable functions.

An initial condition of the form

$$x(0) = x^0 \in R^n \tag{2}$$

is usually associated to (1), but other types of auxiliary conditions can be imposed.

In a recent paper [5], we have considered the case when the operator V is acting on the space $L^\infty([0, T], R^n)$, and is given by

$$(Vx)(t) = x(t) + g(x(t-h)), \qquad h > 0. \tag{3}$$

More precisely, it has been shown in [5] that the Eq. (1) with V given by (3), has a local solution (for which $x(t) + g(x(t-h))$ is an absolutely continuous function), satisfying an initial functional condition of the form

$$x(s) \in L^\infty([-h, 0], R^n), \qquad s \in [-h, 0], \tag{4}$$

provided $g: R^n \to R^n$ is a homeomorphism of R^n, while

$$W: L^\infty([0, T], R^n) \to L^\infty([0, T], R^n)$$

is a continuous operator taking bounded sets into bounded sets.

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In this paper we will prove that a similar result to the one mentioned above can be obtained in case of variable delays. The operator V in (1) will be now of the form

$$(Vx)(t) = x(t) + g(x(\alpha(t))), \qquad t \in [0, T], \tag{5}$$

where $\alpha(t)$ is a scalar (real-valued) function satisfying $0 \leqslant \alpha(t) \leqslant t$ for $t \in [0, T]$, with $\alpha(0) = 0$.

Unlike the case when V is given by (3), the method of integration by steps cannot be used in case (5).

2. STATEMENT OF LOCAL EXISTENCE RESULT.

We shall discuss the existence problem for Eq. (1), with V given by the formula (5). The following assumptions will be made on the data.

(A_1) $g : \mathscr{C}([0, T], R^n) \to \mathscr{C}([0, T], R^n)$ is a contraction operator on this space:

$$|g(x) - g(y)|_C \leqslant \lambda |x - y|_C, \tag{6}$$

with $\lambda \in (0, 1)$, and any $x, y \in \mathscr{C}$.

(A_2) $W : \mathscr{C}([0, T], R^n) \to \mathscr{C}([0, T], R^n)$ is a continuous causal operator, taking bounded sets of $\mathscr{C}$ into bounded sets.

THEOREM 1. *Let us consider the neutral functional differential equation* (1), *with V given by the formula* (5), *under initial condition* (2). *Assume* (A_1) *and* (A_2) *are satisfied, while α is a continuous real valued function such that $\alpha(0) = 0$ and $0 \leqslant \alpha(t) \leqslant t$, $t \in [0, T]$.*

Then, there exists a solution $x = x(t)$ of the problem (1), (2), *defined on an interval $[0, \delta] \subset [0, T]$ such that $x(t) + g(x(\alpha(t)))$ is continuously differentiable.*

The proof will be conducted in the next section. We want now to point out the fact that, without loss of generality, we can assume $g(x^0) = \theta =$ the null element of $\mathscr{C}$. Indeed, if this condition is not satisfied by $g(x)$, $x \in \mathscr{C}$, then we can substitute to $g(x)$ the function $\bar{g}(x) = g(x) - g(x^0)$. Because $g(x)$ appears in (1) under the differentiation sign, it is obvious that equation (1) is the same. Moreover, if g satisfies condition (6), so does $\bar{g}$.

Therefore, we can discuss the existence problem under the extra condition $g(x^0) = \theta$, which does not represent a restriction.

In order to prove Theorem 1, it is useful to transform the Eq. (1), with initial condition (2), into a single functional equation. One obtains by integrating both sides from 0 to $t > 0$,

$$x(t) + g(x(\alpha(t))) = x^0 + \int_0^t (Wx)(s)\,ds, \tag{7}$$

for as long as $x(t)$ is defined ($t > 0$). The two conditions $\alpha(0) = 0$ and $g(x^0) = \theta$ lead exactly to the Eq. (7). Of course, one can differentiate both sides of (7) with respect to t on any interval $[0, t_0]$ on which we know there exists a continuous solution to (7). It is obvious that (7) implies (2), letting $t = 0$.

Consequently, we will have to prove the existence of a continuous solution to the Eq. (7), defined on some interval $[0, \delta]$, $\delta > 0$.

3. PROOF OF THEOREM 1

Let us denote

$$(Ux)(t) = x^0 + \int_0^t (Wx)(s)\,ds, \qquad t \in [0, T], \tag{8}$$

which makes sense for any $x \in \mathscr{C}([0, T], R^n)$. Then, we can rewrite the Eq. (7) in the equivalent form

$$(Vx)(t) = (Ux)(t). \tag{9}$$

If we succeed to show that V has an inverse on $\mathscr{C}([0, T], R^n)$, continuous and causal, then equation (9) becomes

$$x(t) = V^{-1}((Ux)(t)), \tag{10}$$

which represents the usual form (in view of applying Schauder fixed-point theorem) for equations with causal operators (see [3]).

Therefore, the first step in the proof of Theorem 1 is to show that the operator V defined by (5) is onto $\mathscr{C}$, while its inverse V^{-1} does exist, is continuous and causal on $\mathscr{C}$. In other words, the map

$$x(t) \to (Vx)(t) \tag{11}$$

is a homeomorphism of the space $\mathscr{C}$. This will be possible, relatively easy, if we take into account a result due to T. A. Burton [2].

Based on our assumptions (A_1) and (A_2), with $\alpha = \alpha(t)$ as described above, it is obvious that the map (11) is continuous. We have to show that it is onto $\mathscr{C}([0, T], R^n)$, and it is one to one. To this purpose, let us deal with the functional equation (this time the term "functional" is understood in the traditional sense, as it appears, for instance, in the book [1] by J. Aczel)

$$x(t) + g(x(\alpha(t))) = f(t), \tag{12}$$

in the space $\mathscr{C}([0, T], R^n)$, i.e., $(Vx)(t) = f(t)$. The Eq. (12) can be also written as

$$x(t) = -g(x(\alpha(t))) + f(t) = (Tx)(t). \tag{13}$$

Since g is by assumption (A_1) a contraction on $\mathscr{C}([0, T], R^n)$, from (13) we see that the map T is also a contraction. Hence, Eq. (13) has a *unique* solution $x(t) \in \mathscr{C}([0, T], R^n)$, for each $f(t) \in \mathscr{C}([0, T], R^n)$. This says that V maps $\mathscr{C}$ onto $\mathscr{C}$, and for each $f(t) \in \mathscr{C}([0, T], R^n)$, there is only one solution $x(t)$ of (13). In other words, the map (11) has an inverse V^{-1} on $\mathscr{C}$. The Burton's result now applies and we conclude that V^{-1} is continuous. Hence, the map (13) defines a homeomorphism of the space $\mathscr{C}([0, T], R^n)$.

Returning to the Eq. (10), we notice that the product $V^{-1}U$ is a continuous causal operator on $\mathscr{C}$. It is actually a compact operator because U is compact on $\mathscr{C}$. This can be easily seen if we rely on (A_2). If $B \subset \mathscr{C}$ is a bounded set, then $WB \subset \mathscr{C}$ is also bounded. Therefore, there is $M > 0$, such that $y \in WB$ implies

$$|y(t) - y(s)| \leqslant M|t - s|,$$

which means the equicontinuity of the functions belonging to WB. Ascoli–Arzelà criterion of compactness applies, telling us that $U : \mathscr{C} \to \mathscr{C}$ is compact. The product $V^{-1}U$, of a continuous operator and a compact one, is obviously compact (takes bounded sets into relatively compact sets).

We also point out the fact that the operator $V^{-1}U$ has the fixed initial value property. For $t = 0$, $(Ux)(0) = x^0$ for each $x \in \mathscr{C}$, which means that $V^{-1}(x^0)$ is the fixed initial value for $V^{-1}U$.

All conditions required by Theorem 3.4.1 in [3] are satisfied by the operator $V^{-1}U$, which implies the existence of a solution to Eq. (10), on some interval $[0, \delta] \subset [0, T]$. But (10) is equivalent to (9), which in turn is equivalent to (1), (2).

This ends the proof of Theorem 1.

Remark 1. In [2], T. A. Burton shows that the homeomorphism is valid even in case the contraction map is replaced by a weak type of map, called "large contraction".

Remark 2. We have chosen the space $\mathscr{C}([0, T], R^n)$ as underlying space in solving Eq. (1), under initial condition (2). The result in [3] we have applied above (Theorem 3.4.1) covers not only the space $\mathscr{C}$, but also the spaces L^p, $1 \leqslant p < \infty$. One could adapt the above result to the situation when the underlying space is an L^p-space. Then, the equation (1) will be satisfied only almost everywhere on $[0, \delta]$, with $x(t) + g(x(\alpha(t))$ an absolutely continuous function.

We leave the reader the task to formulate and prove results similar to Theorem 1, in case of L^p-spaces. We notice the fact that the property of fixed initial value is not required (does not necessarily make sense).

Some results regarding L^p-spaces are contained in the paper [8] by the author and M. Mahdavi, where $V = I + C$, with C a compact operator. The continuous case is dealt with in [7].

4. A RESULT OF UNIQUENESS

Under the assumptions of Theorem 1, the uniqueness of the solution may not be true. A very simple example can be constructed as follows: Choose $g(x) = \theta$, and $(Wx)(t) = f(t, x(t))$, with f continuous and such that $\dot{x}(t) = f(t, x(t))$ is deprived of uniqueness.

Therefore, extra assumptions must be made on (1), in order to obtain uniqueness of the solution for the problem (1), (2). The nature of the problem suggests that a Lipschitz type condition on W may be sufficient to assure uniqueness.

(A_3) $W: \mathscr{C}([0, T], R^n) \to \mathscr{C}([0, T], R^n)$ satisfies the generalized Lipschitz condition

$$|(Wx)(t) - (Wy)(t)| \leqslant \mu(t) \sup_{0 \leqslant s \leqslant t} |x(s) - y(s)|, \tag{14}$$

for $t \in [0, T]$, and all $x, y \in \mathscr{C}$, with $\mu(t)$ nonnegative on $[0, T]$.

One can read from (14) that W is causal, continuous, and takes bounded sets of $\mathscr{C}$ into bounded sets. Hence, (A_3) implies (A_2).

It is also easy to see that $\mu(t)$ must be a nondecreasing function: $\mu(t_1) \leqslant \mu(t_2)$ for $t_1, t_2 \in [0, T]$, $t_1 \leqslant t_2$.

THEOREM 2. *Under assumptions* (A_1) *and* (A_3), *the solution of the initial value problem* (1), (2), *with* V *given by* (5) *is unique.*

Proof. Let $x(t)$ and $y(t)$ be two solutions of (1), (2), defined on some interval $[0, \delta]$. Then

$$x(t) - y(t) + g(x(\alpha(t))) - g(y(\alpha(t))) = \int_0^t [(Wx)(s) - (Wy)(s)]\, ds,$$

on the interval $[0, \delta]$. The above equality implies

$$|x(t) - y(t)| - |g(x(\alpha(t))) - g(y(\alpha(t)))| \leqslant \int_0^t |(Wx)(s) - (Wy)(s)|\, ds,$$

and on behalf of our assumptions we derive

$$|x(t) - y(t)| - \lambda |x(\alpha(t)) - y(\alpha(t))| \leqslant \int_0^t \mu(s) \sup_{0 \leqslant u \leqslant s} |x(u) - y(u)|\, ds.$$

This inequality can be strengthened, to obtain

$$|x(t) - y(t)| \leqslant \lambda \sup_{0 \leqslant s \leqslant t} |x(s) - y(s)| + \int_0^t \mu(s) \sup_{0 \leqslant u \leqslant s} |x(u) - y(u)|\, ds.$$

Since the right hand side of the inequality is nondecreasing in t, we can write

$$\sup_{0 \leqslant s \leqslant t} |x(s) - y(s)| \leqslant \lambda \sup_{0 \leqslant s \leqslant t} |x(s) - y(s)| + \int_0^t \mu(s) \sup_{0 \leqslant u \leqslant s} |x(u) - y(u)|\, ds$$

or, denoting $z(t) = \sup |x(s) - y(s)|$, $0 \leqslant s \leqslant t$,

$$(1 - \lambda)\, z(t) \leqslant \int_0^t \mu(s)\, z(s)\, ds, \qquad t \in [0, \delta],$$

which implies for each $\varepsilon > 0$

$$z(t) < \varepsilon + (1 - \lambda)^{-1} \mu(\delta) \int_0^t z(s)\, ds, \qquad t \in [0, \delta]. \tag{15}$$

The inequality (15) is of Gronwall type, which means

$$z(t) \leqslant \varepsilon \exp\{(1 - \lambda)^{-1} \mu(\delta)\, t\}, \qquad t \in [0, \delta]. \tag{16}$$

The arbitrariness of $\varepsilon > 0$ implies $z(t) = 0$ on $[0, \delta]$, which is equivalent to $x(t) = y(t)$ on the same interval.

This ends the proof of Theorem 2.

Remark 1. The result in Theorem 2 could be improved somewhat, substituting to (14) the condition

$$|(Wx)(t) - (Wy)(t)| \leqslant \mu(t)\, h(\sup_{0 \leqslant s \leqslant t} |x(s) - y(s)|),$$

where $h(\cdot)$ is a function of Osgood type:

$$\int_{0+} \frac{ds}{h(s)} = +\infty.$$

Then, instead of Gronwall's inequality, one has to use Bihari's integral inequality. Or, using the same argument as in the proof of Osgood's uniqueness theorem for ordinary differential equations.

Remark 2. Under hypotheses (A_1) and (A_3), the following scheme of successive approximations (rather theoretical than practical) is convergent.

One constructs the sequence $\{x^{(m)}(t); m \geqslant 1\}$ by the iteration procedure

$$x^{(m+1)}(t) + g(x^{(m+1)}(\alpha(t))) = x^0 + \int_0^t (Wx^{(m)})(s)\, ds, \qquad m \geqslant 1,$$

starting with an arbitrary $x^{(1)}(t)$. Because g is a contraction map on $\mathscr{C}([0, T], R^n)$, the above equation uniquely defines $x^{(m+1)}(t)$, as soon as $x^{(m)}(t)$ is known.

Using the same procedure as in the proof of Theorem 2, one finds the following recurrence inequality:

$$|x^{(m+1)}(t) - x^{(m)}(t)| \leqslant (1-\lambda)^{-1} \int_0^t \mu(s) \sup_{0 \leqslant u \leqslant s} |x^{(m)}(u) - x^{(m-1)}(u)|\, ds,$$

which can be handled in the usual way to prove the convergence of the sequence $\{x^{(m)}(t); m \geqslant 1\}$. The convergence is uniform on the whole interval $[0, T]$, a feature that leads to the global existence of the solution for (1), (2).

5. AN ALTERNATE APPROACH

Our discussion of Eq. (1) was related only to the case when V is given by (5). If we notice that (1) is equivalent to the equation

$$(Vx)(t) = c + \int_0^t (Wx)(s)\, ds, \qquad t \in [0, T], \tag{17}$$

with $c \in R^n$ an arbitrary vector, then in case V has an inverse V^{-1}, we can rewrite (17) in the form

$$x(t) = V^{-1}\left(c + \int_0^t (Wx)(s)\, ds\right). \tag{18}$$

We notice the fact that V does not have to be necessarily causal. What is really important now is to have a causal inverse for V, so that (18) is an equation with causal operator, of the type investigated in [3], [11].

In case V was defined by (5), or by (3) in the paper [5], we have been able to "construct" the inverse of V. In general, the existence of V^{-1}, as well as its causality or continuity could be taken as a hypothesis.

If we assume that W satisfies condition (A_2), while V is such that it admits an inverse V^{-1} on $\mathscr{C}([0, T], R^n)$, causal and continuous, then Eq. (18) is locally solvable (i.e., it has a solution in some space $\mathscr{C}([0, \delta], R^n)$) for every $c \in R^n$. This is again a consequence of Theorem 3.4.1 in [3].

In order to obtain a result of global existence for (18), it appears reasonable to assume a certain growth condition on V^{-1}. Such a very special condition could be, for instance, the boundedness of V^{-1} on the whole space $\mathscr{C}$, which makes sense only in case V is a homeomorphism between a bounded subset of $\mathscr{C}$ and $\mathscr{C}$. Such an example, in case of the space $\mathscr{C}([0, T], R)$ is given by $(Vx)(t) = \tan x(t)$, where $x(t)$ is an arbitrary element of $\mathscr{C}([0, T], R)$, satisfying $|x(t)| < \frac{\pi}{2}$. The inverse operator is, in this case, $(V^{-1}y)(t) = \tan^{-1} y(t)$, taking for the function $\tan^{-1}$ the value between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

If we admit the boundedness of V^{-1} on the whole $\mathscr{C}$, then, with W satisfying (A_2), the Eq. (18) has a global solution (i.e., defined on $[0, T]$).

Indeed, under condition (A_2), the operator

$$x(t) \to c + \int_0^t (Wx)(s)\, ds$$

is continuous and compact. Since V^{-1} is assumed continuous on $\mathscr{C}$, the operator in the right hand side of (18) is continuous and compact on $\mathscr{C}([0, T], R^n)$, and all that remains to be shown is V^{-1} takes a bounded, closed and convex set of $\mathscr{C}$ into itself.

Let $B = V^{-1}\mathscr{C}$, which is by hypothesis a bounded set in $\mathscr{C}$. Denote by $\tilde{B}$ a ball of sufficiently large radius, such that $B \subset \tilde{B}$. Then $V^{-1}\tilde{B} \subset V^{-1}\mathscr{C} = B \subset \tilde{B}$. Hence, Schauder fixed point applies and we obtain the existence of a solution to (18), in the space $\mathscr{C}([0, T], R^n)$.

Summarizing the above discussion, we can state the following result:

THEOREM 3. *Consider the Eq.* (1), *with V such that it is a homeomorphism between a bounded subset of $\mathscr{C}([0, T], R^n)$ and $\mathscr{C}([0, T], R^n)$. Assume further that V^{-1} is a causal operator on $\mathscr{C}([0, T], R^n)$. The operator W is supposed to satisfy the condition A_2 above.*

Then, there exists a solution of (1) *in $\mathscr{C}([0, T], R^n)$. More precisely, both Eqs.* (17) *and* (18), *which are equivalent to* (1), *have a solution in $\mathscr{C}([0, T], R^n)$ for every $c \in R^n$.*

In concluding this paper, let us mention that a comprehensive account on neutral functional differential equations, in an early stage and somewhat different definition, is given in J. K. Hale's book [9]. See also J. K. Hale and K. R. Meyer [10], with particular regard to the linear case. A more recent contribution is due to S. M. Verduyn Lunel and D. V. Yakubovich [12], also dealing with the linear case.

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